General Description of Defining Euler Equations with many choice and state variables.

Consider the following constrained Bellman Equation:

$$
\begin{align}
W(\theta_t) =& \max_{x_t} F(\theta_t, x_t) + \beta W(\theta_{t+1}) \tag{1} \\
\text{Subjet To:}& \tag{2} \\
\theta_{t+1} =& G(\theta_t, x_t) \tag{3}
\end{align}
$$

Where $\theta_t$ is a $m \times 1$ vector of state variables and $x_t$ is a $k \times 1$ vector of choice variables.

The resulting optimality conditions in matrix form are

$$
\begin{align}
[0] =& \underset{k\times 1}{\frac{\partial F(\theta_t, x_t)}{\partial x_t}} + \beta \underset{k\times m}{\frac{\partial G(\theta_t, x_t)}{\partial x_t}} \cdot \underset{m\times 1}{\frac{\partial W(\theta_{t+1})}{\partial \theta_{t+1}}} \tag{4} \\
[0] =& \vec{f}_{x_t} + \beta B_t \cdot \vec{W}_{\theta_{t+1}} \tag{5}
\end{align}
$$

Similarly the envelope condations can be written as:

$$
\begin{align}
\underset{m\times 1}{\frac{\partial W(\theta_t)}{\partial \theta_t}} =& \underset{m\times 1}{\frac{\partial F(\theta_t, x_t)}{\partial \theta_t}} + \beta \underset{m\times m}{\frac{\partial G(\theta_t, x_t)}{\partial \theta_t}} \cdot \underset{m\times 1}{\frac{\partial W(\theta_{t+1})}{\partial \theta_{t+1}}} \tag{6} \\
\vec{W}_{\theta_t} =& \vec{f}_{\theta_t} + \beta A_t \cdot \vec{W}_{\theta_{t+1}} \tag{7}
\end{align}
$$

If $A_t$ is invertible, it gives the iteration condition

$$
\vec{W}_{\theta_{t+1}} = \beta^{-1} A_t^{-1} \cdot \left[ \vec{W}_{\theta_t} - \vec{f}_{\theta_t} \right] \tag{8}
$$

If $A_t$ is not invertible, you don't have a solution leading to the standard euler equation. But, assuming it is, we can now begin solving for the Euler Equation.

The basic approach is to choose enough extra

The first problem we need to address is dimensionality conserns