Test Title

William King

March 6, 2021

1 Introduction

Introduction goes here. Don't include much yet.

2 Laws of Motion

In this model there are two types of entities subject to laws of motion; i.e. constellation-level satellite stocks and debris. These laws are the foundations to the results found in sections 3 and 4.2, and are crucial elements of the models presented in sections sections 5.1 and 5.3.

2.1 Satellite Stocks

Each constellation consists of a number of satellites in orbit, controlled by the same operator and operated for the same purpose. Satellites can be destroyed by collisions with other satellites or debris. Of course, satellite stocks can be increased by launching more satellites. Assuming satellites live indefinitely, these facts give us the following law of motion for each constellation i.

$$S^i_{t+1} = \left(1 - l^i(\{s^j_t\}, D_t)\right) s^i_t + x^i_t \tag{1}$$

Where $l^i(\cdot)$ represents the rate at which satellites are destroyed by collisions. Note that it is reasonable to assume that the loss of satellites to collisions should be increasing in the level of debris: $\frac{\partial l^i}{\partial D_t} > 0$.

2.1.1 Collision Efficiencies

As demonstrated by **reiland2020**, there are constellation designs by which an operator can minimize the risk of intra-constellation collisions. I assume that when designing a constellation, the operator chooses to minimize collision risks, and as a result, there is a greater relative risk of inter-constellation collision.

It is reasonable to ask why operators would not use the same techniques to reduce inter-constellation collision risks? While some of the steps could be taken, a fundamental issue arises in that constellations are operated for different purposes and require different orbital properties. This coordination is also complicated by the fact that many of the constellations that will add to the overall risk have not been concieved by their designers yet.

Consequent to these reasons, I believe the loss function l^i should have the following properties related to satellite stocks.

$$\frac{\partial l^i}{\partial s^k_t} > 0 \quad \forall k \in \{1, \ldots, N) \tag{2}$$

$$\frac{\partial l^i}{\partial s^j_t} > \frac{\partial l^i}{\partial s^i_t} \quad \forall j \neq i \tag{3}$$

2.2 Debris

Debris is generated by various processes, including:

- Naturally occuring debris
- Satellite launches, operations, failures, or intentional destruction.
- Collisions between satellites
- Collisions between satellites and debris
- Collisions between debris

Debris leaves orbit when atmospheric drag slows it down enough to reenter the atmosphere.

These effects can be represented by the following general law of motion.

$$D_{t+1} = (1-\delta)D_t + g(D_t) + \gamma(\{s^j_t\}, D_t) + \Gamma(\{x^j_t\}) \tag{4}$$

I formulate this more specifically as:

$$D_{t+1} = (1-\delta)D_t + g(D_t) + \sum_{i=1}^{N} \gamma l^i(\{s^j_t\}, D_t) + \Gamma \sum_{j=1}^{n} \{x^j_t\} \tag{5}$$

where Γ, γ represent the debris generated by each launch and collision respectively, while $\delta, g(\cdot)$ represent the decay rate of debris and the autocatalysis[1] of debris generation.

3 Kessler Syndrome

In **Kessler1978** the authors described and forecasted what has come to be known as "kessler syndrome", where debris collides with itself in such a way that the overall debris level grows exponentially. A few methods have been used to model this behavior in the economics literature.

The first one I want to explain was developed by **Adilov2018**. They characterize kessler syndrome as the point in time at which an orbit is unusable as each satellite launched will be destroyed within a single time period. In my notation, this is that $l^i(\{s^j_t\}, D_t) = 1$. The benefit of this approach is that it is algebraically simple. It was used in this role to show that firms will stop launching before orbits are rendered physically useless. Unfortunately, it does not convey the original intent of "kessler syndrome", i.e. a runaway pollution effect, but instead corresponds to the end result of kessler syndrome.

The second common definition of "kessler syndrome" is due to **RaoRondina**. They define it in terms of a "kessler region", the set of satellite stocks and the debris level such that:

$$\kappa = \left\{ \{s^j_t\}, D_t : \lim_{k\to\infty} D_{t+k}\left(\{s^j_{t+k-1}\}, D_{t+k-1}, \{x^j\}\right) = \infty \right\} \tag{6}$$

[1] Using terminology from (RaoRondina2020).

3.1 My approach to kessler syndrome

I propose to analyze kessler syndrome in a slightly more restricted fashion than **RaoRondina**. I would define the ϵ-kessler region as:

$$\kappa = \Big\{ \{s_t^j\}, D_t : \forall k \geq 0, D_{t+k+1} - D_{t+k} \geq \epsilon > 0 \Big\} \tag{7}$$

It is easily shown that this criteria is sufficient to guarantee Rao and Rondina's criteria. It has three primary benefits:

- The ϵ-kessler region can be numerically described within bounded state spaces.
- In a computational model, as most models of any complexity will be, you cannot check for divergence numerically. The condition given is a basic guarantee of the divergent behavior that is required for Kessler Syndrome and acknowledges computational limitations.
- Finally, a slow divergence is no divergence in the grand scheme of things. It is possible to have a mathematically divergent function, but one that is so slow, there is no noticable degree of debris growth before Sol enters a red giant phase. In this specification, it is possible to choose ϵ such that the divergent behaviors identified have an impact on a meaningful timescale.

There is at least one issue with this definition of ϵ-kessler regions. Let's define a "proto-kesslerian" region as the stock and debris levels such that:

$$\kappa = \Big\{ \{s_t^j\}, D_t : D_{t+1} - D_t \geq \epsilon > 0 \Big\} \tag{8}$$

It may be, under certain situations, the case that optimal launch rates cycle along with debris and stock levels, leading to a cycle in and out of the proto-kesslerian regions. This is an issue because, assumning a stable cycle, Rao's definition of the kessler region would capture this behavior, but the ϵ-kessler definition would not. I believe, but have not verified, that some choices of ϵ, although permitting cycles, would relegate them to levels with minimal economic impact.

This leads to the important question of "What makes a good value of ϵ?" One method, in the spirit of **Adilov2018**, is to choose a change in debris, $D_{t+1} - D_t$, such that the loss of satellites in periods $t+1$ to $t+k$ is increased by or to a certain percentage, say 50%. I've put very little thought into addressing this general question so far, and need to analyze the implications of different choice rules.

4 Motion Results

The following are two results due to the laws of motion presented.

4.1 Kessler Regions

Given the definition of kessler syndrome and the law of debris above, we can now explicitly describe the proto-kessler region.

$$\epsilon < -\delta D_t + g(D_t) + \gamma \sum_{j=1}^{n} l^i(\{s_t^j\}, D_t) + \Gamma \sum_{j=1}^{n} \{x_t^j\} \tag{9}$$

As being in the proto-kessler region is a prerequesit to being in the kessler region, we see that the kessler region depends on the collision rates of the constellation operators.

Although this is a straightforward result, I have not found it in any of the models I've examined so far. I suspect it will impact optimal pigouvian taxation, but of course, I need to verify this.

4.2 Survival Analysis

In his dissertation **RaoDissertation** briefly examines the "survival rates" of a satellite constellation. I've applied this to my model and extended the results.

The survival rate for a constellation i is defined as $R_i = 1 - l^i(\cdot)$, the proportion of satellites- that were not lost (degraded nor destroyed) between period t and $t+1$. Thus the marginal survival rate represents the additional loss of satellites due to a slightly larger constellation or fleet stock.

Let $S_t = \sum_{j=1}^{n} s_t^j$. Then the survival rates for a constellation and for society's fleet are respectively defined as:

$$R_i = \frac{s_{t+1}^i - x_t^i}{s_t^i} = 1 - l^i(s_t^i, S_t, D_t) \tag{10}$$

$$R = \frac{S_{t+1} - X_t}{S_t} = \frac{\sum_{i=1}^{N} s_t^i [1 - l^i(s_t^i, S_t, D_t)]}{S_t} \tag{11}$$

In this case, the fleet survival rate eq. (11), represents the proportion of satellites- in period $t+1$ that survived from period t.

The marginal survival rates when a given constellation i changes size are:

$$\frac{\partial R_i}{\partial s_t^i} = -\left(\frac{\partial l^i}{\partial s_t^i} + \frac{\partial l^i}{\partial S_t} \frac{\partial S_t}{\partial s_t^i} \right) = -\frac{\partial l^i}{\partial s_t^i} - \frac{\partial l^i}{\partial S_t} \tag{12}$$

$$\begin{aligned} \frac{\partial R}{\partial s_t^i} &= \frac{S_t \sum_{i=1}^{N} \left([1 - l^i(s_t^i, S_t, D_t)] + s_t^i [-\frac{\partial l^i}{\partial s_t^i} - \frac{\partial l^i}{\partial S_t} \frac{\partial S_t}{\partial s_t^i}] \right) - \left(\sum_{i=1}^{N} s_t^i [1 - l^i(s_t^i, S_t, D_t)] \right)}{(S_t)^2} \\ &= \sum_{i=1}^{N} \left[\frac{R_i}{S_t} \right] - \frac{R}{S_t} + \sum_{i=1}^{N} \frac{s_t^i}{S_t} \frac{\partial R_i}{\partial s_t^i} \end{aligned} \tag{13}$$

Note that $\sum_{i=1}^{N} \frac{s_t^i}{S_t} \frac{\partial R_i}{\partial s_t^i}$ is the weighted, average marginal survival rate- across constellation operators. The derivation of eq. (13) is in Appendix 8.2.1. Direct comparison between the marginal survival rates of an individual operator and the social planner's fleet cannot proceed further without specifying the functional loss forms $l^i(\cdot)$ and specifying which firm will be compared to society. In spite of this, conditions on the average effects can be developed as follows.

The marginal survival rate of the fleet is less than the weighted, arithmetic mean of marginal survival rates- of the constellations when:

$$\sum_{i=1}^{N}\left[\frac{R_i}{S_t}\right] - \frac{R}{S_t} + \sum_{i=1}^{N}\frac{s_t^i}{S_t}\frac{\partial R_i}{\partial s_t^i} \leq \sum_{i=1}^{N}\frac{s_t^i}{S_t}\frac{\partial R_i}{\partial s_t^i} \tag{14}$$

$$\sum_{i=1}^{N} R_i - R \leq 0 \tag{15}$$

$$\sum_{i=1}^{N}[1 - l^i(s_t^i, S_t, D_t)] - \sum_{i=1}^{N} s_t^i[1 - l^i(s_t^i, S_t, D_t)] \leq 0 \tag{16}$$

$$\sum_{i=1}^{N}(1 - s_t^i)[1 - l^i(s_t^i, S_t, D_t)] \leq 0 \tag{17}$$

which is true if every constellation has at least one satellite. As any constellation of interest has at least one satellite and $\frac{\partial R_i}{\partial s_t^i} < 0$ from the assumption on collision mechanics that $\frac{dl^i}{ds_t^i} > 0$, we conclude that the marginal survival rate of the entire satellite fleet is lower than the weighted arithmetic mean of marginal survival rates across constellations. Note that it is possible for some constellations to have a lower marginal survival rate than the fleet, but the survival rate for many operators must be higher than the societal rate. Consequently, we would expect many operators to underestimate the impact of their behaviors on others if they use their own observed or expected risk factors to estimate the risk they impose on others.

5 Model

5.1 Constellation Operator's Program

Introduction goes here

5.2 testing

This is a subsection of the introduction.

5.3 Social Planner's Program

Introduction goes here

5.4 testing

This is a subsection of the introduction.

6 Computation

No work has been done here so far.

7 Assumptions and Caveats

I hope to write a section clearly explaining assumptions, caveats, and shortcomings here.

8 Appedicies

8.1 Mathematical Notation

Needs completed.

8.2 Derivations

8.2.1 Marginal Survival Rates

Derivations related to the marginal survival rate analysis.

$$\frac{\partial R_i}{\partial s_t^i} = -\left(\frac{\partial l^i}{\partial s_t^i} + \frac{\partial l^i}{\partial S_t}\frac{\partial S_t}{\partial s_t^i}\right) = -\frac{\partial l^i}{\partial s_t^i} - \frac{\partial l^i}{\partial S_t}$$

$$\begin{aligned}
\frac{\partial R}{\partial s_t^i} &= \frac{S_t \sum_{i=1}^{N}\left([1 - l^i(s_t^i, S_t, D_t)] + s_t^i[-\frac{\partial l^i}{\partial s_t^i} - \frac{\partial l^i}{\partial S_t}\frac{\partial S_t}{\partial s_t^i}]\right) - \left(\sum_{i=1}^{N} s_t^i[1 - l^i(s_t^i, S_t, D_t)]\right)}{(S_t)^2} \\
&= \sum_{i=1}^{N}\left[\frac{S_t[1 - l^i(s_t^i, S_t, D_t)]}{(S_t)^2} - \frac{s_t^i[1 - l^i(s_t^i, S_t, D_t)]}{(S_t)^2}\right] + \sum_{i=1}^{N}\frac{s_t^i S_t[-\frac{\partial l^i}{\partial s_t^i} - \frac{\partial l^i}{\partial S_t}]}{(S_t)^2} \\
&= \sum_{i=1}^{N}\left[\frac{S_t - s_t^i}{(S_t)^2}[1 - l^i(s_t^i, S_t, D_t)]\right] + \sum_{i=1}^{N}\frac{s_t^i}{S_t}\frac{\partial R_i}{\partial s_t^i} \\
&= \sum_{i=1}^{N}\left[\frac{1}{S_t}[1 - l^i(s_t^i, S_t, D_t)]\right] - \frac{R}{S_t} + \sum_{i=1}^{N}\frac{s_t^i}{S_t}\frac{\partial R_i}{\partial s_t^i} \\
&= \sum_{i=1}^{N}\left[\frac{R_i}{S_t}\right] - \frac{R}{S_t} + \sum_{i=1}^{N}\frac{s_t^i}{S_t}\frac{\partial R_i}{\partial s_t^i}
\end{aligned}$$