Rewrite orbits

Constellation Operator's Program

Given $\{\{S_t\}^j\}, \{X_t\}^{-i}$ and D_t

2N States

$$V^i\left(\{S_t^j\}_{j=1}^n, \{X_t^j\}_{i\neq j}, D_t\right) = \max_{X_t^i} \left\{ U^i\left(\{S_t^j\}_{j=1}^n, D_t\right) - F(X_t^i) + \beta V^i\left(\{S_{t+1}^j\}^n, \{X_{t+1}\}, D\right)\right.$$

Sub to

Debris

$$D_{t+1} = (1-\delta)D_t + g(D_t) + \gamma \sum_{j=1}^n \ell^j\left(\{S_t^j\}_{j=1}^n, D_t\right) + \Gamma \sum_{j=1}^{\Lambda} X_t^j$$

Vector Notation

$$= \sim\sim + \vec{\gamma}^T \vec{\ell}\left(\{S_t^j\}_{j=1}^n, D_t\right) + \vec{\Gamma}^T \cdot \vec{X}_t$$

Satellite types

Launch technologies.

Stocks $S_{t+1}^i = \left(1 - \ell^i\left(\{S_t^j\}_{j=1}^n, D_t\right)\right) S_t^i + X_t^i \quad \forall i$

Vector Notation

$$\vec{S}_{t+1} = \underset{mat}{Dg}\left(\overset{L}{1 - \ell^i}(\vec{S}_t, D_t)\right)\vec{S}_t + \vec{X}_t$$

$$\vec{S}_t \geq [0]$$

Conditions

$$\frac{\partial \ell^i}{\partial S_t^j} > \frac{\partial \ell^i}{\partial S_t^i} \geq 0 \quad \text{Collision efficiencies}$$

$$\frac{\partial \ell^i}{\partial D_t} > 0$$

Constellation Operator's Euler Equation

$$V^i(\vec{s}_t, \vec{x}_t^{\neg i}, D_t) = \underset{x_t^i}{\mathrm{Max}} \; u^i(\vec{s}, D_t) - F^i(x_t^i) + \beta V^i(\vec{s}_{t+1}, \vec{x}_{t+1}^{\neg i}, D_{t+1})$$

$$0 = 0 - \frac{\partial F^i}{\partial x_t^i} + \beta \left[\nabla_{(\vec{s}_{t+1}, \vec{x}_{t+1}^{\neg i}, D_{t+1})} V^i \right]^T \cdot \nabla_{x_t^i} \begin{bmatrix} \vec{s}_{t+1} \\ \vec{x}_{t+1}^{\neg i} \\ D_{t+1} \end{bmatrix}$$

Optimality Condition

$$\frac{1}{\beta} \frac{\partial F^i}{\partial x_t^i} = \nabla_{(\underbrace{\quad}_{t+1})} V^i(\underbrace{\quad}_{t+1})^T \cdot \nabla_{x_t^i} \begin{bmatrix} \vec{s}_{t+1} \\ \vec{x}_{t+1}^{\neg i} \\ D_{t+1} \end{bmatrix}$$

Note that $\vec{x}_{t+1}^{\neg i}(\vec{s}_{t+1}, D_{t+1})$, thus

$$\frac{\partial \vec{x}_{t+1}^{j}}{\partial x_t^i} = \overset{\text{Vec}}{\nabla_{\vec{s}_{t+1}} \vec{x}_{t+1}^{j}} \cdot \nabla_{x_t^i} \vec{s}_{t+1} + \frac{\partial D_{t+1}}{\partial x_t^i} \frac{\partial \vec{x}_{t+1}^{j}}{\partial D_{t+1}} \quad \forall j \neq i$$

$$\frac{\nabla \vec{x}_{t+1}^{\neg i}}{\partial x_t^i} = \overset{\text{matrix}}{\nabla_{\vec{s}_{t+1}} \vec{x}^{\neg i}} \cdot \nabla_{x_t^i} \vec{s}_{t+1} + \nabla_{D_{t+1}} \vec{x}_{t+1}^{\neg i} \cdot \frac{\partial D_{t+1}}{\partial x_t^i}$$

The optimality Condition Can be Iterated 2n-1 times forward to establish

Envelope Conditions

~~n ~~ Operators

$\vec{S}_t : n \times 1$

$\vec{X}_t^{\neg i} : (n-1) \times 1$

$D_t : 1$

thus $(\vec{S}_t, \vec{X}_t^{\neg i}, D_t)$ has $2n$ terms.

$$\underset{1 \times 2n}{\nabla_{(\vec{S}_t, \vec{X}_t^{\neg i}, D_t)} V^i} = \underset{1 \times 2n}{\nabla_{(\vec{S}_t, \vec{X}_t^{\neg i}, D_t)} u^i} - 0$$

$$+ \beta \Big[\underset{\langle 1 \times n \rangle}{\nabla_{\vec{S}_{t+1}} V^{i\,T}} \cdot \underset{\langle n \times 2n \rangle}{\nabla_{(\vec{S}_t, \vec{X}_t^{\neg i}, D_t)} \vec{S}_{t+1}}$$

$$+ \underset{\langle 1 \times (n-1) \rangle}{\nabla_{\vec{X}_{t+1}^{\neg i}} V^{i\,T}} \cdot \underset{\langle (n-1) \times 2n \rangle}{\nabla_{(\sim_t)} \vec{X}_{t+1}^{\neg i}}$$

$$+ \underset{\langle 1 \times 1 \rangle}{\left(\frac{\partial V^i}{\partial D_{t+1}}\right)} \cdot \underset{\langle 1 \times 2n \rangle}{\nabla_{(\sim_t)} D_{t+1}} \Big]$$

$$= \underset{1 \times 2n}{\nabla_{(\vec{S}_{t+1}, \vec{X}_{t+1}^{\neg i}, D_t)} V^{i\,T}} \cdot \underset{2n \times 2n}{\nabla_{(\vec{S}_t, \vec{X}_t^{\neg i}, D_t)} (\vec{S}_{t+1}, \vec{X}_{t+1}^{\neg i}, D_{t+1})}$$

That is what worries me.

So, We need to Iterate the Envelope condition $2n-1$ Periods forward, In order to Substitute

The Lifetime Reward Description is:

Social Planners Problem

$$W(\vec{s}_t, D_t) = \max_{\vec{x}_t} \left\{ \sum_{i=1}^{n} \left(u^i(\vec{s}_t, D_t) - F(x_t^i) \right) + \beta W(\vec{s}_{t+1}, D_{t+1}) \right\}$$

Subject to the same Debris & Stocks situation

Optimality Conditions

$\vec{s}_t : n \times 1$

$$0 = - \sum_{j=1}^{n} \frac{\partial F}{\partial x_t^j} + \beta \left[\nabla_{\vec{s}_{t+1}, D_{t+1}} W_t \cdot \nabla_{\vec{x}_t} \begin{bmatrix} \vec{s}_{t+1} \\ D_{t+1} \end{bmatrix} \right]$$

Envelope Conditions

$$\underset{1 \times (n+1)}{\nabla_{(\vec{s}_t, D_t)} W} = \sum_{j=1}^{n} \underset{1 \times (n+1)}{\nabla_{(\vec{s}_t, D_t)} u^j} - 0 + \beta \underset{1 \times (n+1)}{\nabla_{(\vec{s}_{t+1}, D_{t+1})} W^T} \cdot \underset{(n+1) \times (n+1)}{\nabla_{(\vec{s}_t, D_t)} \begin{bmatrix} \vec{s}_{t+1} \\ D_{t+1} \end{bmatrix}}$$

Discussion of dimensionality.

m: Number of Gridpoints
n: Number of Operators

Discretization Dimensions

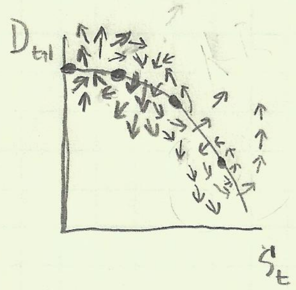
Social Planner: m^{n+1} : Gridpoints for each operator & Debris level

Operator's Problem $m^{2n+(1-1)}$: ~~you have~~ SP + Launch plans

So VFI is out...

Options
- Endogenous Grid methods ← Requires calibration
- DL Approach. ← might not Require calibration.

Possible
Kessler Syndrome Phase diagrams



Proposed Spec

$$l^i = \exp\left(-\sum_{j=1}^{n} \eta s_t^j - (\alpha - \eta) s_t^i - \varphi D_t \right)$$ ← memoryless property

$$u^i(s_t^i) = \pi s_t^i$$

$$g(D_t) = D_t^{\,g}, \quad g \in [1, 2]$$

$$\delta \in \{0.01, 0.1, 0.2, 0.5\}$$

100 yr, 10 yr, 5 yr, 1/2 yr ← Corresponding to a life of δ Before decaying

$\alpha, \eta, \varphi, \Gamma, \gamma$: No Idea

Survival Ratios

$$R = \frac{\sum_{j=1}^{n} s_t^j \left(1 - \ell^j(\vec{S}_t, D_t)\right)}{\sum_{j=1}^{n} s_t^j} = \frac{\sum_{j=1}^{n} s_t^j R_j}{\sum_{j=1}^{n} s_t^j}$$

$$\frac{\partial R}{\partial s_t^i} = \frac{1}{(\sum s_t^j)^2}\left[\left(\sum_{j=1}^{n} s_t^j\right)\left(\sum_{j=1}^{n} \frac{\partial}{\partial s_t^i} s_t^j R_j\right) - \left(\sum_{j=1}^{n} s_t^j R_j\right)(1)\right]$$

$$= \frac{1}{\sum_{j=1}^{n} s_t^j}\left[\left(\sum_{j=1}^{n} \frac{\partial}{\partial s_t^i} s_t^j R_j\right) - R\right]$$

Thus

$$R + \left(\sum_{j=1}^{n} s_t^j\right)\frac{\partial R}{\partial s_t^i} = \sum_{j=1}^{n} \frac{\partial}{\partial s_t^i} s_t^j R = \left(\sum_{j \neq i} s_j \frac{\partial R_j}{\partial s_t^i}\right) + \left(s_i \frac{\partial R_i}{\partial s_t^i} + (1) R_i\right)$$

Let $S_t = \sum_{j=1}^{m} s_t^j$

$$R + \left(\sum_{j=1}^{n} s_t^j\right)\frac{\partial R}{\partial s_t^i} = \sum_{j=1}^{n}\left(s_j \frac{\partial R_j}{\partial s_t^i}\right) + R_i$$

Note.
If $\frac{\partial \ell^i}{\partial s_t^i} > 0$
then
$R_i = 1 - \ell^i$
$\frac{\partial R_i}{\partial s_t^i} = -\frac{\partial \ell^i}{\partial s_t^i} < 0$

$$\boxed{\frac{R - R_i}{S_t} + \frac{\partial R}{\partial s_t^i} = \sum_{j=1}^{n} \frac{s_t^j}{S_t}\frac{\partial R_j}{\partial s_t^i} < 0}$$

So, is $(R - R_i \lesseqgtr 0)$? or is it indeterminant?

$S_t \geq 0$ By definition

Also

$$R - R_i = \frac{\sum_{j=1}^{n} s_t^j (1 - \ell^j)}{S_t} - \frac{S_t s_t^i (1 - \ell^i)}{S_t}$$

$$\left(\frac{1}{S_t}\right)\sum_{j \neq i} s_t^j (1 - \ell^j) + (1 - S_t)\, s_t^i (1 - \ell^i)$$

$$\frac{R - R_i}{S_t} + \frac{\partial R}{\partial s_t^i} < 0$$

$$1 - \frac{R_i}{R} < - \frac{\partial R}{\partial s_t^i} \frac{S_t}{R}$$

$$\frac{R_i}{R} - 1 > \frac{S_t}{R} \frac{\partial R}{\partial s_t^i}$$

Does this make Sense as an elasticity?

Not Realy, as it is working with a ~~Specific~~ change to a specific Constellation.

$$\frac{R - R_i}{S_t} < - \frac{\partial R}{\partial s_t^i}$$

$$\left(\frac{s_t^i}{S_t}\right)\left(\frac{R - R_i}{R}\right) < - \frac{s_t^i}{R} \frac{\partial R}{\partial s_t^i}$$

$$\left(\frac{s_t^i}{S_t}\right)\left(\frac{R_i}{R} - 1\right) > \frac{\partial R}{\partial s_t^i} \frac{s_t^i}{R}$$

~~Elasticity of Loss~~

Elasticity of Loss w/Respect to size of constellation "i"

Not Sure how useful this Is.

On the whole, Very commonly Less than zero, I Believe.

~~Appears to be always~~

This can probably be Split into A "Substitution" and an "Income" effect

$$\left(\frac{s_t^i}{\cancel{S_t}}\right)\left(\frac{(1-\ell^i)\cancel{X}}{(\Sigma\, 1-\ell^i)} - 1\right)$$

$$0 \geq s_t^i \left(\frac{-\sum_{k \neq i} 1-\ell^k}{\sum_{j=1}^{n} 1-\ell^j}\right)$$

Double Check

Wrote down R incorrectly

In a Stochastic model, this might be boundable.